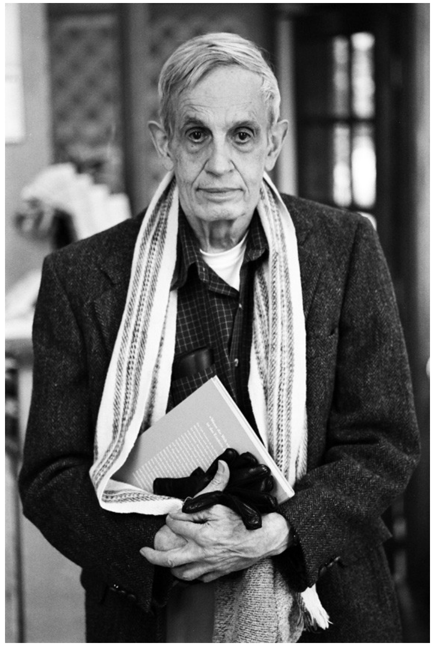
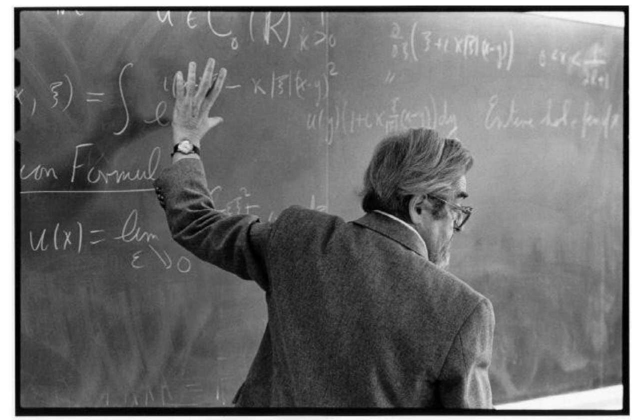
Mathematical contributions of John F. Nash





Abel Prize 2015 to the American mathematicians
John F. Nash, Jr. and **Louis Nirenberg**
''for striking and seminal contributions to the theory of nonlinear partial differential equations and its applications to geometric analysis''

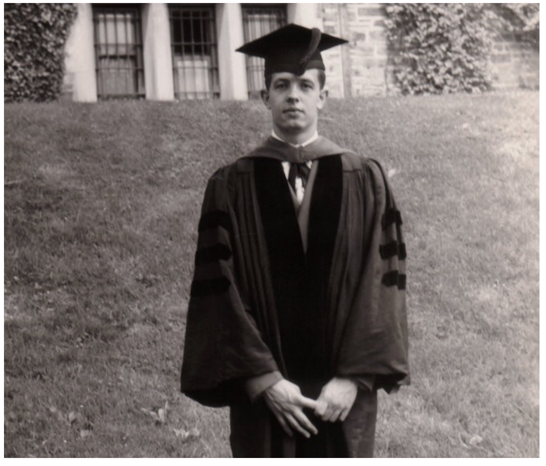
Xavier Cabré, ICREA Research Professor at the UPC
09/11/2015 SCE-SCM

John F. Nash, Jr.

- Born June 13, 1928. Bluefield, West Virginia, U.S.
- Died May 23, 2015 (aged 86). New Jersey, U.S.

- Institutions:
  Massachusetts Institute of Technology
  Princeton University

- Notable awards:
  John von Neumann Theory Prize (1978)
  Nobel Memorial Prize in Economic Sciences (1994)
  Abel Prize (2015)



John F. Nash Jr. at his Princeton graduation in 1950, when he received his doctorate.

Main mathematical contributions of John F. Nash:

- **Game Theory:**

  Nash equilibra and their existence

- **Partial Differential Equations:**

  De Giorgi-Nash theorem

- **Riemannian Geometry:**

  Nash embedding theorem

- **Analysis:**

  Nash-Moser implicit function theorem

Author Citations for John Forbes Nash Jr.

John Forbes Nash Jr. is cited 1611 times by 1828 authors

in the MR Citation Database

Most Cited Publications	
Citations	**Publication**
346	**MR0043432 (13,261g)** Nash, John Non-cooperative games. *Ann. of Math. (2)* **54,** (1951). 286–295. (Reviewer: D. Gale) 90.0X
296	**MR0100158 (20 #6592)** Nash, J. Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.* **80** 1958 931–954. (Reviewer: C. B. Morrey Jr.) 35.00
251	**MR0031701 (11,192c)** Nash, John F., Jr. Equilibrium points in n-person games. *Proc. Nat. Acad. Sci. U. S. A.* **36,** (1950). 48–49. (Reviewer: L. Törnqvist) 90.0X
204	**MR0075639 (17,782b)** Nash, John The imbedding problem for Riemannian manifolds. *Ann. of Math. (2)* **63** (1956), 20–63. (Reviewer: J. Schwartz) 53.1X
170	**MR0035977 (12,40a)** Nash, John F., Jr. The bargaining problem. *Econometrica* **18,** (1950). 155–162. (Reviewer: K. J. Arrow) 90.0X
96	**MR0149094 (26 #6590)** Nash, John Le problème de Cauchy pour les équations différentielles d'un fluide général. (French) *Bull. Soc. Math. France* **90** 1962 487–497. (Reviewer: M. Schechter) 35.79
63	**MR0065993 (16,515e)** Nash, John C^1 isometric imbeddings. *Ann. of Math. (2)* **60,** (1954). 383–396. (Reviewer: S. Chern) 53.0X
61	**MR1381967 (98f:14011)** Nash, John F., Jr. Arc structure of singularities. A celebration of John F. Nash, Jr. *Duke Math. J. 81 (1995), no. 1,* 31–38 (1996). 14E15
55	**MR0050928 (14,403b)** Nash, John Real algebraic manifolds. *Ann. of Math. (2)* **56,** (1952). 405–421. (Reviewer: W. V. D. Hodge) 14.0X
50	**MR0053471 (14,778i)** Nash, John Two-person cooperative games. *Econometrica* **21,** (1953). 128–140. (Reviewer: D. Gale) 90.0X
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AMERICAN MATHEMATICAL SOCIETY
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Mathematical Reviews

ISSN 2167-5163

University Politicnica de Catalunya

Edit Author Profile

Nash, John Forbes, Jr.

MR Author ID:	**366251**
Earliest Indexed Publication:	1950
Total Publications:	**26**
Total Author/Related Publications:	**54**
Total Citations:	**1611**

⊞ Published as: Nash, J. F. ...

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Co-authors (by number of collaborations)

Hammerstein, Peter   Harsanyi, John C.   Kuhn, Harold W.   Selten, Reinhard   Shapley, Lloyd S.   van Damme, Eric E. C.   Weibull, Jörgen W.

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John Forbes Nash, Jr.

Biography MathSciNet

Ph.D. Princeton University 1950

Dissertation: *Non-Cooperative Games*

Advisor: Albert William Tucker

Student:

Name	School	Year	Descendants
Seth Patinkin	Princeton University	2003	

According to our current on-line database, John Nash, Jr. has 1 student and 1 descendant.
We welcome any additional information.

If you have additional information or corrections regarding this mathematician, please use the update form. To submit students of this mathematician, please use the new data form, noting this mathematician's MGP ID of 18590 for the advisor ID.

De Giorgi-Nash-Moser Theorem: Hölder regularity of solutions of

$$\partial_i(a_{ij}(x)\partial_j u) = f(x)$$

with a_{ij} uniformly elliptic (positive definite matrices) but only bounded and measurable as a function of x in R^n.

- **Nash, J.** *Parabolic equations.* Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 754-758.
- **Nash, J.** *Continuity of solutions of parabolic and elliptic equations.* Amer. J. Math. 80 (1958), 931-954.

"A gold mine", in Nirenberg's words.

Nash work retaken and presented in:

- Fabes, E. B.; Stroock, D. W**.** *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash.* Arch. Rational Mech. Anal. 96 (1986), no. 4, 327-338.

Independently proved by:

- **De Giorgi, Ennio.** *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari.* (Italian) Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 1957 25–43.

and later a new (third) proof by **Jürgen Moser**

There is a striking resemblance on the modeling of

- heat &
- option prices in Finance

In both cases the basic object is the same: "the Laplacian" after

Pierre-Simon, marquis de Laplace (1749-1827)

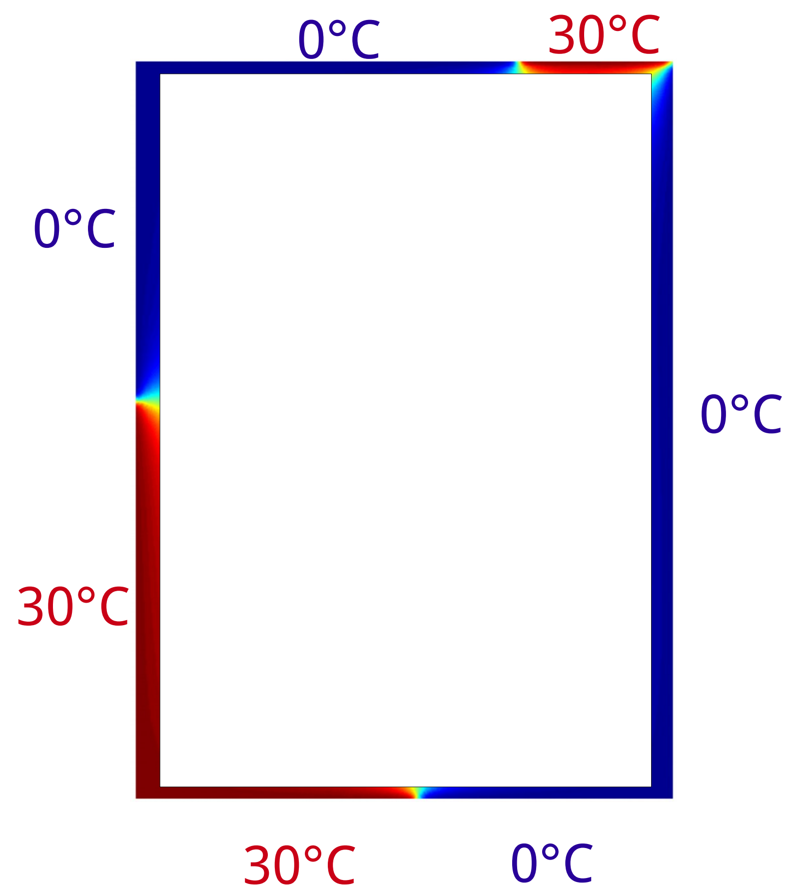
It is responsible for many phenomena in our lives

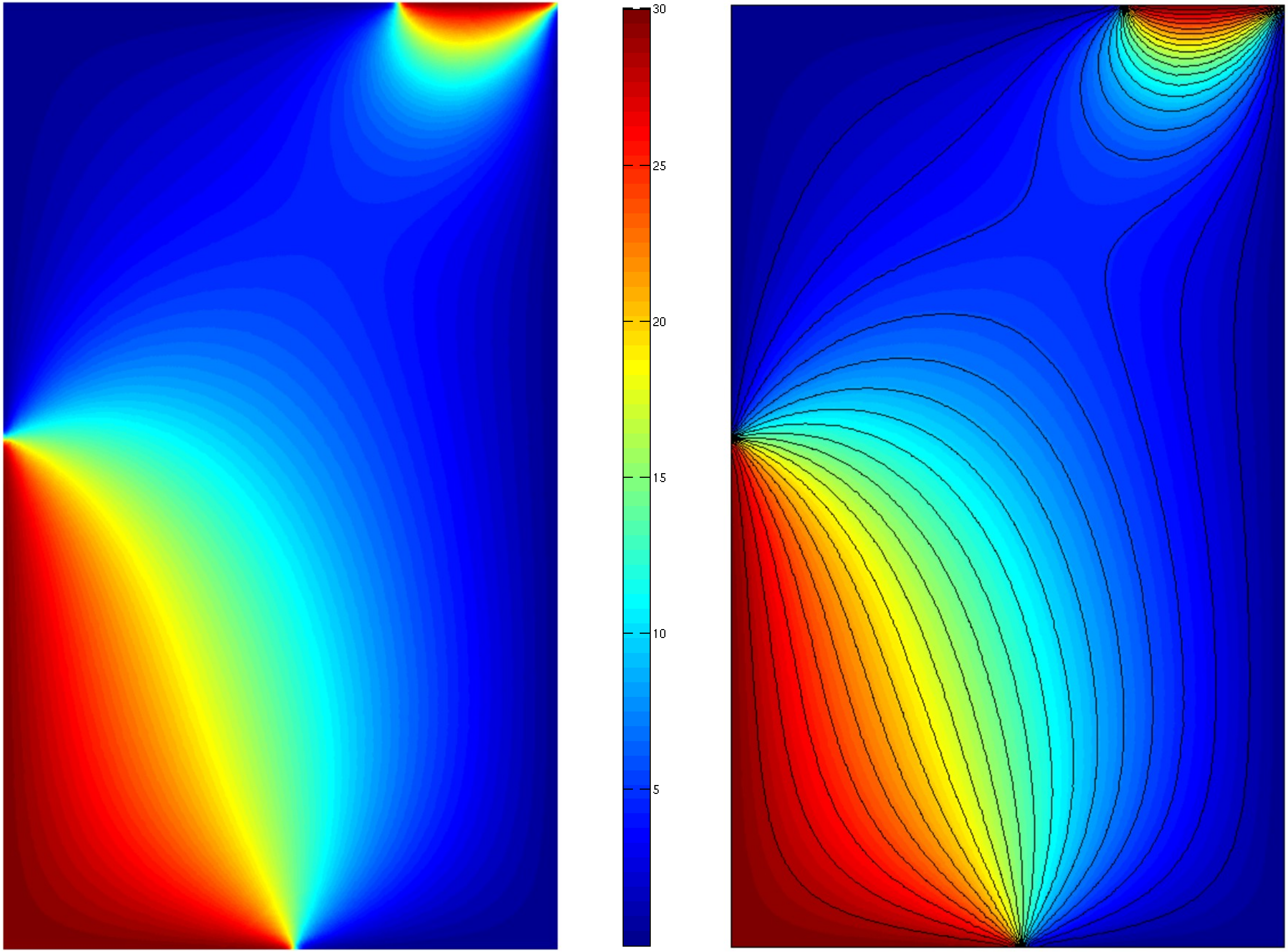
- A first example:

  what is the **temperature** of a certain tile in your living room's floor,

  long after you turn on the wall radiators at 30°C

  while the remaining of the walls are always kept at 0°C ?





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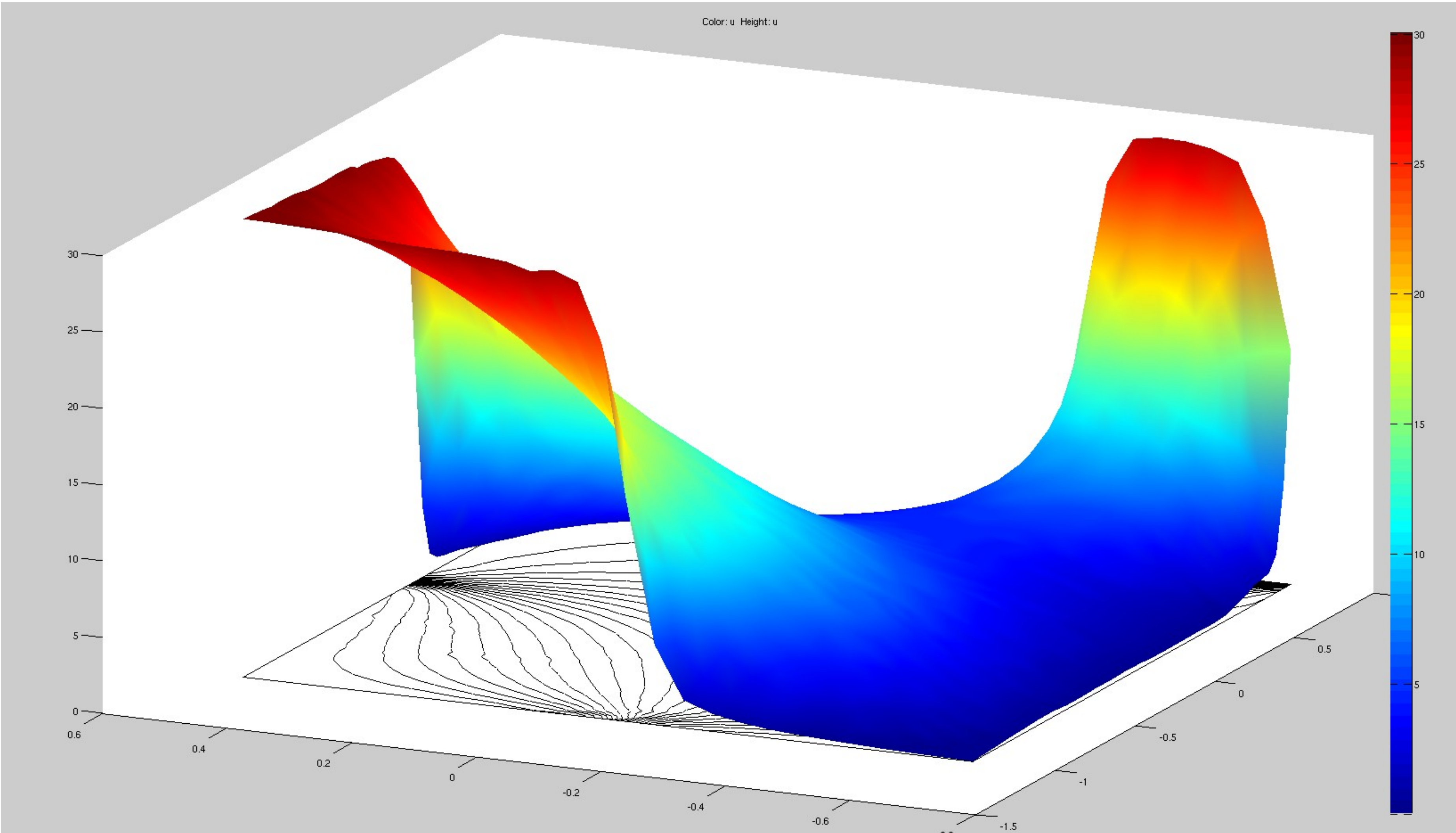
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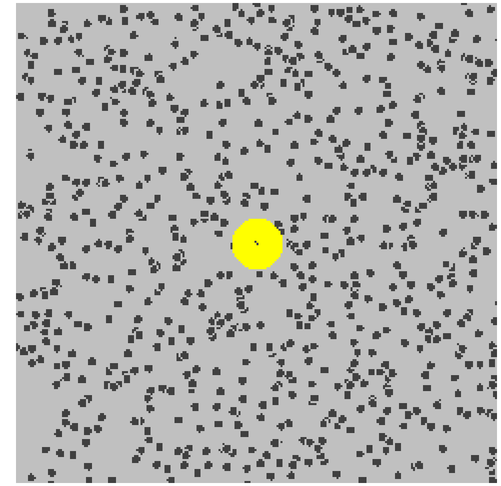
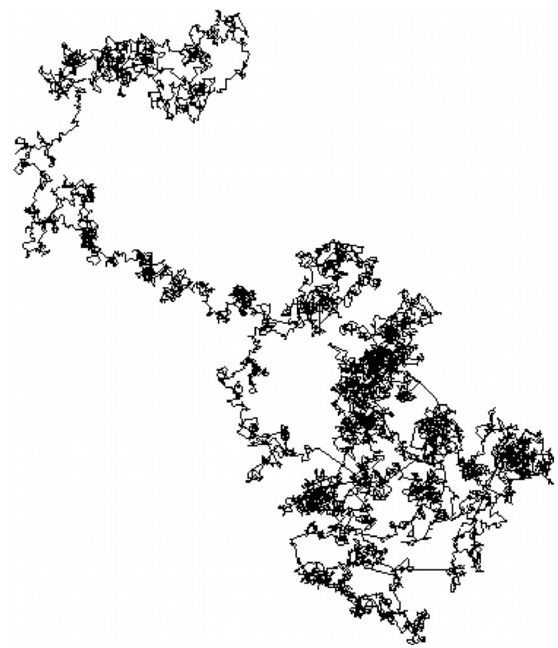
Robert Brown (1773-1858), biologist

Looking through a microscope at pollen grains in water, he noted that the grains moved randomly through the water



BROWNIAN MOTION

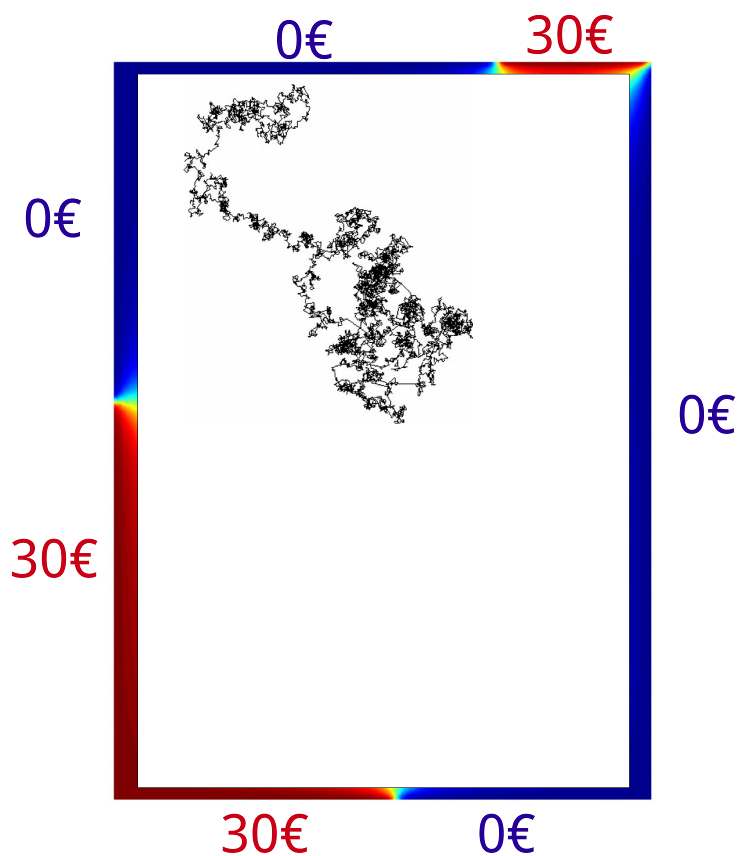


Think also on a large plastic beach ball on the stands of a stadium totally full of people

- **A second example:**

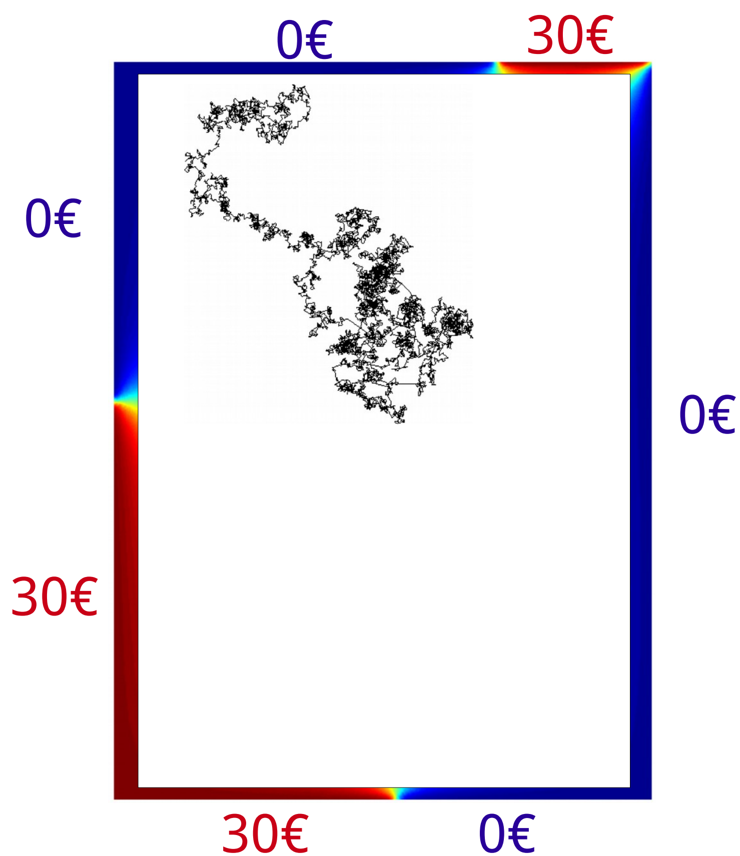
  what is your **expected gain** when,

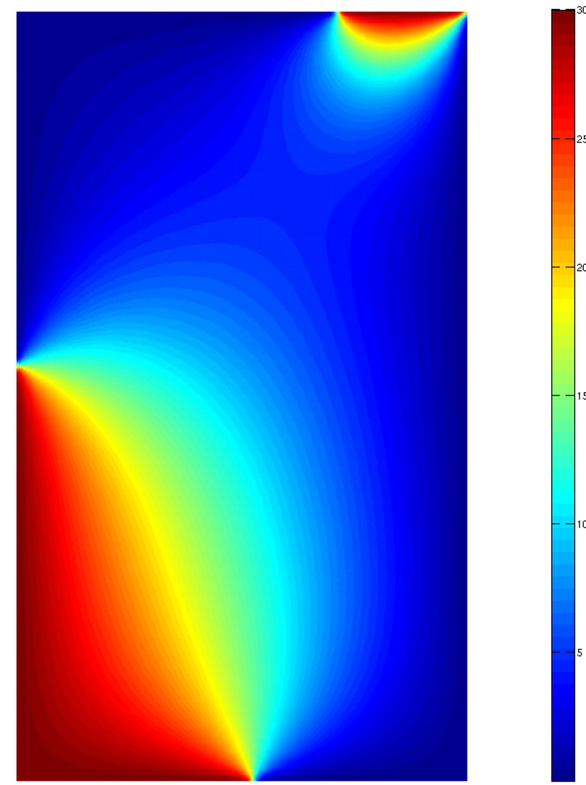
  starting always from the same given tile in your living room, you walk randomly and you get 30€ only when you hit a radiator on the first time that you hit your living room's walls (otherwise you get 0€)?



ANSWER: at every point one has

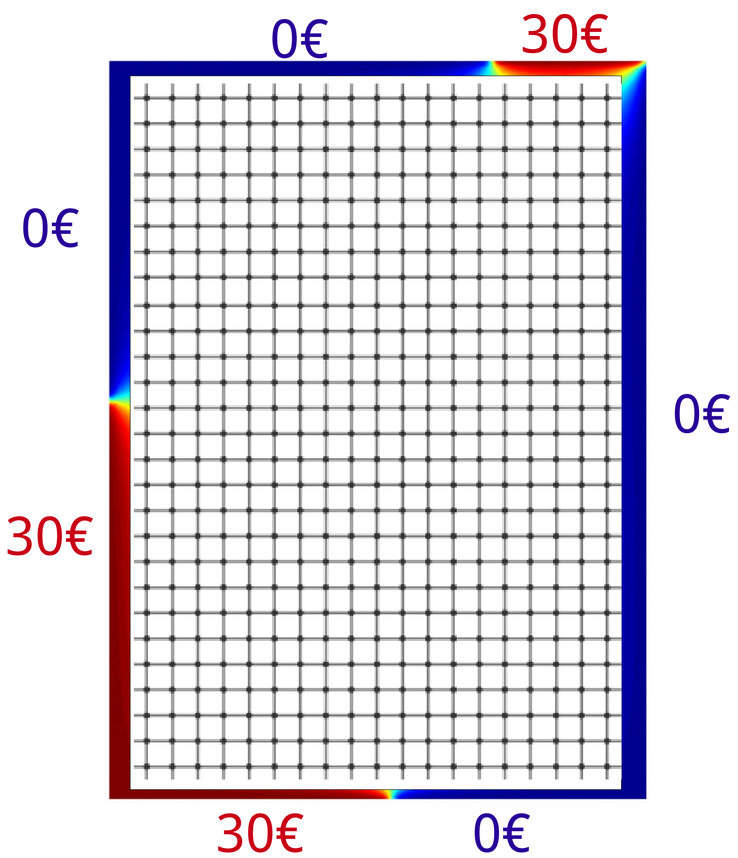
expected gain = temperature !!





How to solve the problem:

- make a squared lattice of very small step-size h
- Move from a point to either East, West, North, or South, each one with probability 1/4



How to solve the problem:

- make a **squared lattice** of very small **step-size h**
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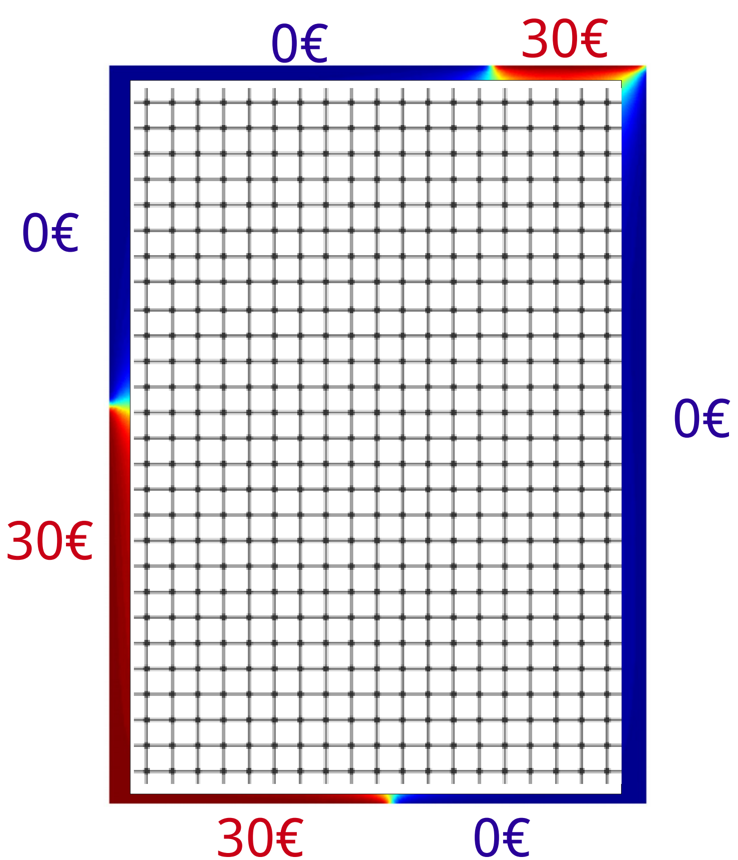
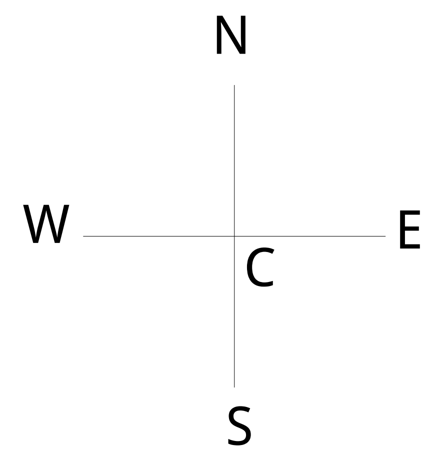




C = starting point of the walk

u(C) = expected gain starting from C

$$u(C) = \frac{1}{4}\{u(E) + u(W) + u(N) + u(S)\}$$

(**average**)

h = step size of the lattice

$$u(C) = \frac{1}{4}\{u(E) + u(W) + u(N) + u(S)\}$$

$$u(x,y) = \frac{1}{4}\{u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)\}$$

$$\frac{u(x+h,y) + u(x-h,y) - 2u(x,y)}{h^2} + \\ + \frac{u(x,y+h) + u(x,y-h) - 2u(x,y)}{h^2} = 0$$

$$\Delta u\,(x,y) = (\,\partial_{xx} u + \partial_{yy} u\,)\,(x,y) = 0$$

The LAPLACIAN of u = 0

- X. Cabré, *Partial differential equations, geometry and stochastic control,* in Catalan. Butl. Soc. Catalana Mat. 15 (2000), 7-27
- X. Cabré, *Elliptic PDEs in Probability and Geometry. Symmetry and regularity of solutions*. Discrete Contin. Dyn. Syst. 20 (2008), 425-457

$$\Delta u = \partial_{xx} u + \partial_{yy} u = 0$$

is called Laplace equation

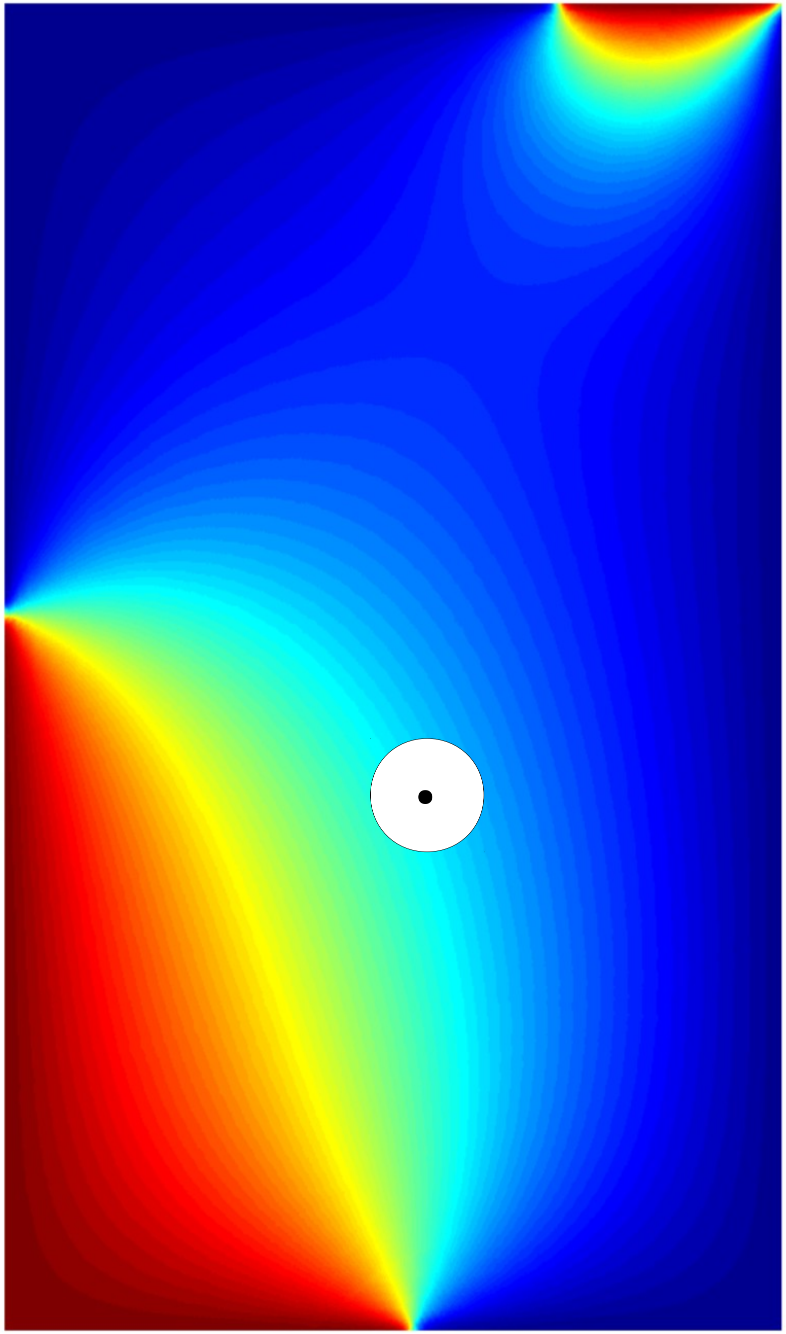
It is a **Partial Differential Equation** (a PDE)
(also called the equations of Mathematical Physics)

Its solutions are called “harmonic functions”. Together with solutions of the heat or diffusion equation

$$\partial_t u - \Delta u = 0$$

(and other equations of the same type), they model:

- heat (Fourier and Einstein)
- option prices in Finance
- gravitational and electric potentials (Laplace)
- densities of biological or chemical species



Harmonic functions are characterized by <u>the mean value property</u> :

The value of the function
at the center of any circle
=
the average of the values
of the function on the circle

OK with HEAT,
and with EXPECTED GAIN !

$$\Delta u = \partial_{xx} u + \partial_{yy} u = 0$$

is called Laplace equation

When the probabilities to move in a certain direction (being at a given point) **depend on the solution u=u(x,y) itself**

(for instance: if I am more risky when I have more money u, or if the heat conductivity properties of a material depend on how hot it is)

Then **the equations become NONLINEAR:**

$$\partial_x\{\, a(x,y,u)\, \partial_x u \,\} + \partial_y\{\, b(x,y,u)\, \partial_y u \,\} = 0$$

and much more difficult to analyze. The main contribution of Nash is to prove the continuity of the solution u in a quantitative way. This is extremely important to be able to analyze and compute numerically the solution.

Partial Differential Equations. Types :

1. Elliptic : Laplace equation: $\Delta u = \partial_{xx} u + \partial_{yy} u = 0$

2. Parabolic :

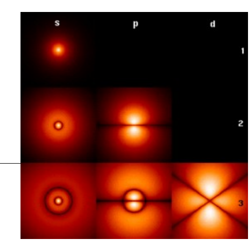
- **Heat** or diffusion equation: $\partial_t u - \Delta u = 0$
- Navier-Stokes (or 1 million \$) equations (incompressible viscous **fluids**)

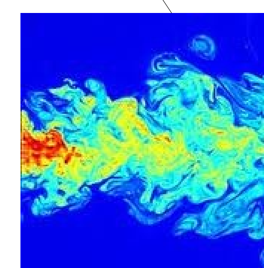
$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot D\mathbf{u} - \nu \Delta \mathbf{u} = -Dp \\ \text{div } \mathbf{u} = 0 \end{cases}$$

3. Hyperbolic :

- **Wave** equation (acoustics, sound-waves) $\partial_{tt} u - \Delta u = 0$

- Schrödinger equation (quantum mechanics) $i\partial_t u + \Delta u = 0$

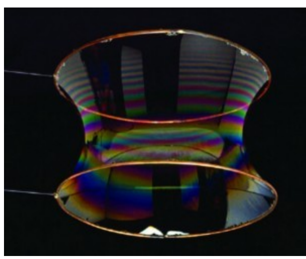




- Euler's equations (incompressible **fluids**)

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot D\mathbf{u} = -Dp \\ \text{div } \mathbf{u} = 0 \end{cases}$$

Some other important PDEs:



a. Linear equations.

1. Laplace's equations: $\Delta u = 0$
2. Helmholtz's equation (involves eigenvalues): $-\Delta u = \lambda u$
3. First-order linear transport equation: $u_t + c\, u_x = 0$
4. Heat or diffusion equation: $u_t - \Delta u = 0$
5. Schrödinger's equation: $i\, u_t + \Delta u = 0$
6. Wave equation: $u_{tt} - c^2 \Delta u = 0$
7. Telegraph equation: $u_{tt} + d\, u_t - u_{xx} = 0$

b. Nonlinear equations.

1. Eikonal equation: $|Du| = 1$
2. Nonlinear Poisson equation: $-\Delta u = f(u)$
3. Burgers' equation: $u_t + u\, u_x = 0$
4. Minimal surface equation: $\operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{1/2}}\right) = 0$
5. Monge-Ampère equation: $\det(D^2 u) = f$
6. Korteweg-deVries equation (KdV): $u_t + u\, u_x + u_{xxx} = 0$
7. Reaction-diffusion equation: $u_t - \Delta u = f(u)$

c. System of partial differential equations.

1. Evolution equation of linear elasticity: $u_{tt} - \mu \Delta u - (\lambda + \mu) D(\operatorname{div} u) = 0$
2. System of conservation laws: $u_t + \operatorname{div} F(u) = 0$
3. Maxwell's equations in vaccum: $\begin{cases} \operatorname{curl} E = -B_t \\ \operatorname{curl} B = \mu_0 \varepsilon_0 E_t \\ \operatorname{div} B = \operatorname{div} E = 0 \end{cases}$
4. Reaction-diffusion system: $u_t - \Delta u = f(u)$
5. Euler's equations for incompressible, inviscid fluid: $\begin{cases} u_t + u \cdot Du = -Dp \\ \operatorname{div} u = 0 \end{cases}$
6. Navier-Stokes equations for incompressible viscous fluid: $\begin{cases} u_t + u \cdot Du - \Delta u = -Dp \\ \operatorname{div} u = 0 \end{cases}$

References:

- (https://www.simonsfoundation.org/science_lives_video/john-nash-2/

  Video of interview to Nash

- http://www.abelprize.no/

  Shortfilms on 2015 Abel Prize Laureates

- http://www.sylvianasar.com/a-beautiful-mind/

  Book on Nash' life and scientific achievements: